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Abstract

This article presents a simple model to determine the location strategies of retail firms that own and operate a number of stores in a geographical market. The firms try to maximize their total profit under competition. Unlike the majority of past studies, the number of stores to be opened is made endogenous by the introduction of fixed costs. Two forms of competition are studied: An entrant against an existing set of firms, and a duopoly in which the firms play a leader-follower game. A novel methodology, in which firms' strategies are represented as continuous functions over the market is developed. This methodology has proven to be quite useful in developing insights as well as detailed solutions that can be implemented by the firms. The analysis has produced several results on the competitive location strategies: First, it is shown that if a late entrant decides to enter a market, she enters at least as strongly as the existing competition. Second, the leader can effectively deter entry even if she is severely cost disadvantaged. However, in some cases the leader is better off by allowing the follower enter the market. Third, the leader may also let a follower enter the market in some situations where she has a cost advantage.

Key words: competitive location, continuous models.

Résumé

Cet article présente un modèle simple pour la détermination des stratégies de localisation de détaillants qui possèdent et administrent plusieurs magasins dans un marché géographique. Les détaillants maximisent leur profit total dans un contexte de compétition. Contrairement à la plupart des études antérieures, le nombre de magasins à ouvrir est endogène en raison de la présence des coûts fixes. On étudie deux formes de compétition : un entrant en compétition avec des détaillants, et un duopole dans lequel les firmes jouent à un jeu de pionnier-suiveur. On présente une méthodologie nouvelle où stratégies des détaillants sont représentées par des fonctions continues sur l'ensemble du marché. Cette méthodologie s'est avérée utile pour le développement de politiques générales et de solutions détaillées. L'analyse a permis de tirer plusieurs conclusions sur les stratégies de localisation compétitive. D'abord, on démontre que si une firme désire s'installer dans un marché existant, elle doit le faire à un niveau au moins aussi élevé que celui de la compétition. Ensuite, un pionnier peut décourager la compétition même s'il souffre d'un désavantage de coût. Toutefois, dans certains cas, le pionnier a avantage à laisser s'installer la compétition. Enfin, le pionnier peut également laisser entrer la compétition même si elle possède un avantage de coût.

Mots-clefs : localisation compétitive, modèles continus.

1 Introduction

Location decisions are of critical importance to firms. Opening, closing, and relocating facilities require intensive planning due to the strategic nature of these decisions. When the facilities do not have physical contact with the customers (such as, plants, distribution centers, or call centers) demand for the products or service can be assumed to be relatively independent of the locations. Therefore, models that minimize the total cost of opening and operating the facilities and servicing the customers have been used extensively in theory and practice. However, the location choices of stores such as supermarkets, bank branches, video clubs, restaurants, and coffee shops, do have a direct impact on demand. Therefore, such decisions should not be made without consideration of demographics, customer choice processes, and the location of the competing stores.

There exists quite an extensive and truly multi-disciplinary literature on competitive location. It ties in several fields including economics, marketing, operations research, and computational geometry. Each of these disciplines takes a different perspective of the problem. For example, the economics literature is mainly concerned with providing insights into competitive location strategies. These models are relatively simple and apply predominantly to the firms that own and operate single stores. In contrast, studies that have appeared in the operations research literature are mostly concerned with providing detailed solutions that can assist the managers' decision making process. Most of the models are defined on networks where consumers and facilities are assumed to be on the nodes or edges. The computational geometry literature deals mainly with algorithms to determine the equilibrium market areas of facilities. Models in operations research and computational geometry provide detailed solutions but little insight. Moreover, they tend to require extensive data and computing times.

While these approaches contain important differences, they share certain common modeling assumptions. First, a majority of the models attempt to maximize the total revenue or market share assuming the number of branches the firms open as exogenous. They also assume that firms are identical in terms of their cost structure. However, having lower operating costs is a competitive advantage that should also be included in the analysis. Second, most of the models entail a very precise representation of the locations where the new stores to be opened. We believe that this is a fixation in pursuit of

theoretical accuracy which has relatively little practical value. Consider a case where a model makes precise suggestions about the locations of stores to be opened. There is quite a good chance that the firm will find some of those suggestions impossible to implement for many reasons that cannot be included in the model. Third, it becomes much harder to derive solutions out of these models as problem size increases. They become intractable either analytically or computationally.

In this article, we propose an alternative methodology for competitive location. It aims to combine the perspectives of economics and operations research. It is based on an approximate rather than a precise representation of the store locations. This approach was introduced to the operations research literature in a seminal paper by Newell (1973) for a different class of problems, and referred as "continuous approximation". This methodology leads to intuitive models without losing much of the substance. It has later been used in production-distribution system design (Geoffrion 1976, 1979), in general optimal market areas (Erlenkotter 1989), and in logistics system analysis (Daganzo 1991). To our knowledge, we are the first to apply it to competitive location.

We consider firms that own and operate multiple stores in a geographical market. Demand at any branch is not only a function of its location but also a function of other branch locations whether owned by the same firm or a competitor. Firms determine location strategies that maximize their total profit under competition. Unlike the majority of past studies, we consider the number of stores to be opened endogenous by the introduction of fixed costs. Hence, our problem resolves the tradeoffs involved in revenue maximization and cost minimization while considering the location strategies of the competitors.

At the outset, our model determines whether a firm should enter a market (or a part of the market) or not. If the answer is affirmative, the model determines the number of stores to be opened and their approximate locations. We leave the determination of exact locations to subsequent analyses, which would most likely include more detailed considerations. As opposed to earlier models, having large number of stores in a market works to our advantage. We will show that the accuracy of our model increase with the number of stores the firms operate.

Before proceeding to our problem description we will give a selective review of the past work in

competitive location. The literature in this area is extensive and our review will focus on the most influential contributions. For detailed reviews the reader may refer to Gabszewicz and Thisse (1986), Ghosh and McLafferty (1987), Eiselt, Laporte, and Thisse (1993), Kilkenny and Thisse (1999), Plastria (2001), and Stole (2001).

The paper by Hotelling (1929) is widely recognized as the study that sets the stage in the economics literature. It considers the competition of two firms on a line segment of unit length. The line can represent a physical or a perceptual space on which the firms locate their facilities or products. This work is just as influential in product differentiation literature as it is in competitive location. Hotelling formulates a two-stage game, where firms choose their location in the first stage and compete on prices in the second stage. However, it is well known that under Hotelling's original assumptions, there may not be a pure-strategy price equilibrium. Several authors have later proposed refinements over the original Hotelling model. These include zero conjectural variation axioms by Eaton (1972), random choice models by Ben-Akiva (1973), McFadden (1973), and Anderson and de Palma (1992), a circular city by Salop (1979), quadratic transport costs by d'Aspremont, Gabszewicz, and Thisse (1979), convex transport costs by Economides (1986), vertical differentiation by Shaked and Sutton (1982), and spatial price discrimination by Lederer and Hurter (1986). Finally, Caplin and Nalebuff (1991) provide fairly general conditions for pure-strategy price equilibrium existence and uniqueness in an oligopoly.

Retail location has also been the object of much research in marketing. Estimating trade areas and market shares of retail branches is central to the store location, relocation or dismantling, and to the management of retail networks. Simple models such as Voronoi tessellations and Reilly's law of retail gravitation have been around for much of the twentieth century (see for example, Ghosh and McLafferty 1987). These methods and their various extensions have been used to demarcate trade areas for retail outlets, especially in the presence of limited data. For example Huff (1964, 1966) suggests a probabilistic choice model that considers distance and the relative attraction of locations. In addition, several computational geometry articles have proposed more comprehensive models using simple Voronoi tessellations. For example, Okabe and Suzuki (1987), Okabe and Aoyagi (1991), and Aoyagi and Okabe (1993) study pure-strategy location strategies of large number of single-branch

firms in a two-dimensional space.

While not directly relevant our model, there exist a large body of literature in economics and marketing that uses econometric models in retail location. McFadden's (1973) multinomial logit (MNL) and Nakanishi and Cooper's (1974) multiplicative competitive interaction (MCI) models have been significantly extended to incorporate several other store characteristics such as store space, merchandise variety and quality, price, and quality of service. Among the more recent econometric models we can mention Ellickson (2000), Davis (2000), and Seim (2001).

The operations research literature is generally concerned with the competitive location on networks. Consumers are assumed to be located either at the nodes or spread uniformly on the edges of the network. Similarly, location choices of firms are restricted to nodes or edges. An important contribution in this area is due by Hakimi (1983) who formulates two kinds of problems under exogenously set prices. In the $(r|p)$-medianoid problem the follower locates r new facilities on a network on which p competitor facilities are already located. In the $(r|p)$-centroid problem the leader locates p facilities so as to minimize the maximum profit of the subsequent r competitor facilities. Hakimi shows that these problems are NP-hard. This seminal work has led to several algorithmic and conceptual extensions. These include maximum capture problems by ReVelle (1986) and Serra and ReVelle (1994), oligopolistic competition with random consumer utility by de Palma et al. (1989), delivered pricing on networks by Lederer and Thisse (1990), Cournot competition by Labbé and Hakimi (1991), incorporation of different choice models by Serra et al. (1999), and a polynomial time algorithm for the $(1|p)$-medianoid by Eiselt and Laporte (2001).

In all of the above models, the number of facilities the competitors can open is exogenously fixed. Models in which the number of facilities is endogenously determined through fixed costs are provided by Dobson and Karmarkar (1987) and Miller, Friesz, and Tobin (1996). Dobson and Karmarkar make various stable set definitions, i.e., the sets of store locations that the leader opens to deter the follower. They present several mixed integer models to find these stable sets. Miller, Friesz, and Tobin describe a set of problems and present formulations to characterize Nash-Cournot and Stackelberg-Nash-Cournot equilibrium locations and production quantities. They use variational inequality techniques to solve these problems. The Dobson and Karmarkar model shares several features with our problem. These

authors have observed that a leader in the market can deter entry by preemptive store opening. However, in their model the firms have identical cost structures. The alternative store locations are also already predetermined and the follower is not allowed to co-locate. Therefore, when the leader's strategies are not constrained, the follower cannot even open a single store that makes profit because the worst the leader can do is to open a store at that node without lowering her profit.

We, on the other hand, treat the consumers as entities continuously spread over the market according to a given density and consider any point in the space as an alternative store location. We also consider relevant costs to be represented as continuous functions defined over the market to account for geographical variations as in Dobson and Karmarkar. However, we allow firms to have different fixed costs. On the revenue side, we assume that the marginal net revenue of firms (marginal revenue less unit cost) is exogenously given. Furthermore, we set these revenues equal without loss of generality.

In the interest of tractability, we impose several restrictions, all of which have been extensively used in the past literature. We consider a bounded market in which firms operate multiple branches. Each branch of a firm displays the same characteristics and therefore, there is horizontal competition between them, i.e., if the branches of the same firm are too close, then they cannibalize each other's demand. We also assume that other store characteristics such as service quality, merchandise assortment, advertising, price, and promotions are exogenously fixed and do not affect the competition. Therefore, consumers are assumed to patronize the closest store. The model could realistically represent many industries given the high degree of franchising and the resulting standardization among sister branches. However, an extension of this model should also consider firms' strategic choices for some of these characteristics.

We study two forms of competition: An entrant competing against an already established set of competitors and a duopoly where the firms play a leader-follower game. We consider both one- and two-dimensional markets but the exposition of continuous approximation and the derivation of the main results are conducted in one-dimensional case for the sake of simplicity. We should note, however, that in certain cases (such as competition on a main road or along a highway) a one-dimensional market could be an accurate representation of reality. Nevertheless, we extend the analysis to two-dimensional case and show that all of the results remain valid under mild conditions.

Our analysis provides several insights into the location strategies of firms under competition. First, we establish that if a late entrant decides to enter a market, she enters at least as strongly as the existing competition. Second, a market leader can effectively deter entry even if she is severely cost disadvantaged. We have found that even if leader's fixed costs are twice those of the follower, she can open enough stores to deter entry and still make a profit. However, in some of those cases the leader is better off by allowing the follower to enter the market. Third, the leader may also let a follower enter and take at least half of the revenues in some cases where she is extremely cost advantaged.

The remainder of this article is organized as follows. In the next section we introduce the additional assumptions, variable definitions, and the methodology on a one-dimensional market. Section 3 analyzes the location strategies of an entrant in a one-dimensional market where competition is already established. A leader-follower game is presented in Section 4. Sections 5 extends the analysis to two-dimensional markets. Concluding remarks and future research directions are provided in Section 6.

2 Continuous Approximation on a Linear Market

We now present our methodology on a linear market where there is only one firm facing no competition, thus laying the groundwork that will be used in subsequent sections. Some of the basic notation and the assumptions are also presented here. Let the consumers be dispersed on the line segment $[0, 1]$ according to a probability density function $f(x)$. Each of them purchases one unit of a product or a service generating the same revenue for the firm. The total attainable revenue is normalized to unity without loss of generality. The firm has n branches located at $x_1, x_2, \ldots, x_n \in [0, 1]$. Consumer $x \in [0, 1]$ derives a utility $u_i(x)$ by patronizing branch i, given as:

$$u_i(x) = a - b|x - x_i|.$$

We assume that a and b are the same for all consumers. A consumer patronizes the facility that provides the largest positive utility. Hence consumer x patronizes store i if

$$u_i(x) \geq u_j(x), \quad \forall j, \text{ and } u_i(x) \geq 0$$

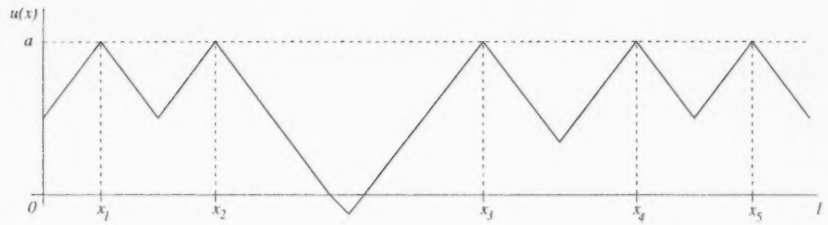

Figure 1: Example of consumer utility in a five-store example.

Let $u(x)$ denote the maximum utility (i.e., intrinsic utility) consumer x derives. Then,

$$u(x) = \max_i u_i(x).$$

Figure 1 depicts the maximum attainable utility for each consumer in an example where the firm has five stores. Note that all consumers, except a group between the second and the third stores, derive positive utility. Therefore, they patronize the closest store, i.e., they are captured by the firm. Let $1_{\{u(x)\geq 0\}}(x)$ be an indicator function denoting whether consumer x is captured or not. Hence, the total revenue can be written as:

$$\int_0^1 1_{\{u(x)\geq 0\}}(x) f(x) dx, \tag{1}$$

where, for each $x \in (x_i, x_{i+1}]$,

$$1_{\{u(x)\geq 0\}}(x) = \begin{cases} 1 & \text{if } x \leq x_i + b/a \\ 1 & \text{if } x \geq x_{i+1} - b/a \\ 0 & \text{if } x_i + b/a < x < x_{i+1} - b/a. \end{cases}$$

To deal with both ends of the line segment we define pseudo-stores at $x_0 = -x_1$ and $x_{n+1} = 2 - x_n$. We now turn our attention to costs in order to determine the total profit. Without loss of generality, we assume that the unit cost of servicing has already been deducted from the unit revenue. Therefore, (1) denotes the net total revenue and the only cost element to be considered is the total fixed cost. Let $K(x)$ denote the fixed cost of opening a branch at location x. Given branch locations $x_1, \ldots, x_n$, the total fixed cost is given as $\sum_j K(x_j)$. Therefore, the problem can be formulated as:

$$\max_{0\leq x_1\leq \ldots \leq x_n \leq 1} \int_0^1 1_{\{u(x)\geq 0\}}(x) f(x) dx - \sum_j K(x_j). \tag{2}$$

One possible way to solve the above problem is to fix n and then devise a search procedure over x_i's. The problem, however, is nonlinear and nonconvex in general. Even though efficient solution algorithms could conceivably be obtained for special cases, we doubt whether the effort is worth the outcome. Instead, we follow an alternative approach based on the approximation of the net total revenue and total cost.

Approximating the net total revenue basically follows a probabilistic analysis of the intrinsic utility, which is essentially deterministic. Consider an arbitrary consumer $x \in (x_i, x_{i+1}]$ for which $z(x)$ is a random variable denoting the minimum distance from x to one of the branches. Clearly, $u(x) = a - bz(x)$ is also a random variable denoting the intrinsic utility. Let $P(x)$ denote the probability that consumer x patronizes one of the stores, i.e., $P(x) = P\{u(x) \geq 0\}$. Therefore, the total revenue is:

$$\int_0^1 P(x)f(x)dx. \tag{3}$$

We now determine the *approximate* distribution of $z(x)$ and hence, that of $u(x)$. Note that, for any $x \in (x_i, x_{i+1}]$,

$$P\{z(x) \geq z\} = 0 \quad \text{if } z \geq (x_{i+1} - x_i)/2.$$

For $z \leq (x_{i+1} - x_i)/2$,

$$P\{z(x) \geq z\} = \frac{\int_{x_i+z}^{x_{i+1}-z} f(x)dx}{\int_{x_i}^{x_{i+1}} f(x)dx}.$$

Assume that consumer density $f(x)$ is a slowly varying function over the interval $(x_i, x_{i+1}]$ so that $f(x) \approx \alpha + \beta x$ is an accurate approximation. Then,

$$P\{z(x) \geq z\} \approx 1 - \frac{2z}{x_{i+1} - x_i}.$$

Hence, $z(x)$'s distribution is approximately uniform on $[0, (x_{i+1} - x_i)/2]$. Similarly $u(x)$ has a uniform distribution between $[a - b(x_{i+1} - x_i)/2, a]$. Finally,

$$P(x) = \begin{cases} 1 & \text{if } x_{i+1} - x_i \leq 2a/b \\ 2a/b(x_{i+1} - x_i) & \text{otherwise.} \end{cases} \tag{4}$$

In order to simplify the expressions we introduce a step function $g_s(x)$ as

$$g_s(x) = x_{i+1} - x_i \text{ for } x \in (x_i, x_{i+1}].$$

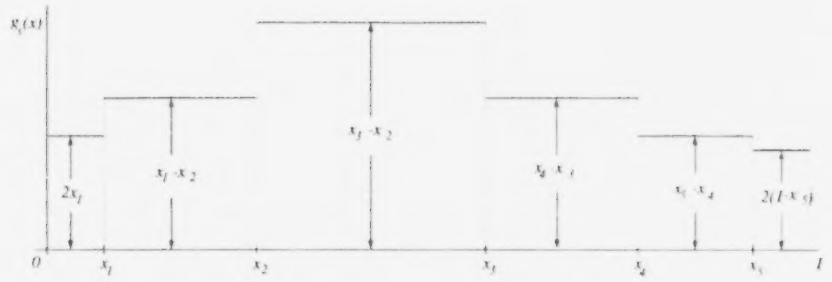

Figure 2: An example of a step function

Note that, no information is lost by the introduction of the step function because there is a one-to-one relationship between the store locations and the step function. An example of a step function is depicted in Figure 2. Hence, for all $x \in [0,1]$,

$$\begin{aligned} z(x) &\sim \text{Uniform}[0, g_s(x)/2], \\ u(x) &\sim \text{Uniform}\,[a - bg_s(x)/2, a], \quad \text{and} \\ P(x) &= \begin{cases} 1 & \text{if } g_s(x) \le 2a/b \\ 2a/bg_s(x) & \text{otherwise.} \end{cases} \end{aligned}$$

As will be seen, the role of this step function goes beyond the simplification of the formulation. We now express the total fixed cost in terms of the step function $g_s(x)$ defined above. First, consider the following identity:

$$\sum_i K(x_i) = \sum_i K(x_i) \left[\int_{(x_i+x_{i-1})/2}^{x_i} \frac{dx}{x_i - x_{i-1}} + \int_{x_i}^{(x_{i+1}+x_i)/2} \frac{2dx}{x_{i+1} - x_i} \right].$$

Note that each of the integrals in the brackets is equal to $1/2$, i.e., we multiply each $K(x_i)$ by one. Using the definition of the step function $g_s(x)$, this expression can be rewritten as:

$$\begin{aligned} \sum_i K(x_i) &= \sum_i K(x_i) \left[\int_{(x_i+x_{i-1})/2}^{x_i} \frac{dx}{g_s(x)} + \int_{x_i}^{(x_{i+1}+x_i)/2} \frac{dx}{g_s(x)} \right] \\ &= \sum_i K(x_i) \int_{(x_i+x_{i-1})/2}^{(x_{i+1}+x_i)/2} \frac{dx}{g_s(x)}. \end{aligned}$$

Similarly, we assume that fixed costs do not change abruptly in close neighborhoods. In effect, $K(x)$ can be perceived as a slowly varying function in x. This assumption allows us to express the total

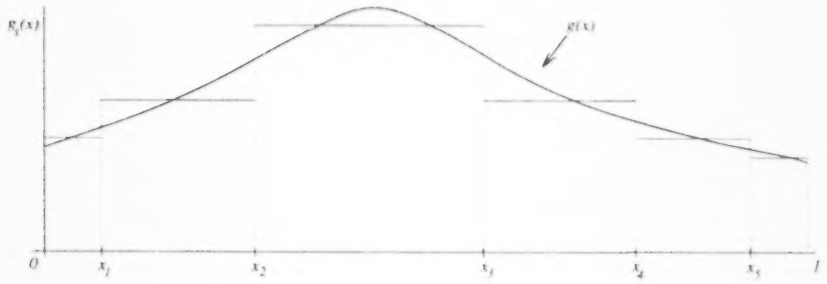

Figure 3: A step function and its continuous approximate

fixed cost as a single integral. Thus,

$$\begin{aligned}\sum_i K(x_i) &\approx \sum_i \int_{(x_i+x_{i-1})/2}^{(x_{i+1}+x_i)/2} \frac{K(x)dx}{g_s(x)} \quad \text{(because } K(x) \text{ is slowly varying)} \\ &= \int_0^1 \frac{K(x)dx}{g_s(x)}.\end{aligned}$$

Given the number of branches, the problem of maximizing the total profit can be rewritten as:

$$\max_{g_s(x)\geq 0} \Pi[g_s(x)] = \int_0^1 \left[P(x)f(x) - \frac{K(x)}{g_s(x)}\right] dx. \tag{5}$$

To formulate the problem, we have assumed that both the consumer density and fixed cost change slowly in close neighborhoods. This assumption is critical for the analysis and it is well justified. We observe that important factors of consumer density and fixed costs, such as demographics and real estate prices, do not display abrupt changes in close neighborhoods. However, solving (5) is akin to solving (2). Therefore, rather than trying to find a step function and hence, the precise branch locations, we first find a smooth function $g(x)$ that solves (5) and then a step function $g_s(x)$ that is consistent with its smooth counterpart can be found by a subsequent analysis. This is the idea behind the continuous approximation approach (see Newell 1973). An example of $g_s(x)$ and $g(x)$ is depicted in Figure 3. Hence, the alternative formulation is:

$$\max_{g(x)\geq 0} \Pi[g(x)] = \int_0^1 \left[P(x)f(x) - \frac{K(x)}{g(x)}\right] dx, \tag{6}$$

where

$$P(x) = \begin{cases} 1 & \text{if } g(x) \leq 2a/b \\ 2a/bg(x) & \text{otherwise.} \end{cases}$$

Similarly,

$$\begin{aligned} z(x) &\sim \text{Uniform}[0, g(x)/2], \\ u(x) &\sim \text{Uniform } [a - bg(x)/2, a]. \end{aligned}$$

Several remarks are in order here: First, the number of branches is endogenous in the new formulation. Therefore, we do not need to perform a search over n to solve the optimization problem. Second, the smooth function approximates the distance between neighbor branches. From this function, we can obtain the approximate store locations and their trade areas. Third, it is clear that such an approximation is quite accurate provided there is large number of branches. This number is somewhat instance specific, but a subsequent analysis will reveal the accuracy of this approach. Note that the accuracy levels of the two approximations are intimately related. An increase in the possible number of branches leads to higher accuracy not only of the second approximation, but also of the first one because the intervals on which $f(x)$ and $K(x)$ need to be slowly varying become smaller. Therefore, our formulation becomes more accurate as the problem size (i.e., the number of branches) increases, leaving the problem complexity virtually unchanged.

Solving (6) is a simple exercise given the problem parameters. Basically, the firm divides the market into two regions, profitable and non-profitable, and opens equidistant stores in the profitable regions. Hence, beyond a threshold value that depends on the firm's fixed cost, consumer density has no impact on the dispersion of the stores. An obvious reason for this is the inelastic demand assumption we have made, i.e., any consumer with a non-negative utility is captured by the firm. A less obvious reason, however, is the absence of any kind of competition. A similar reasoning applies to the fixed cost, i.e., below a certain threshold value that depends on the consumer density, it has no impact on the firm's location strategy. In the forthcoming sections, we show that both consumer density $f(x)$ and fixed cost $K(x)$ have a secondary impact on the firms' location strategies under competition.

3 An Entrant Against an Existing Competition

In this section we analyze the location strategies of a firm planning to open a number of stores in a market that is already being served by other firms. We call the firm an *entrant*, even though in some cases she may decide not to enter the market. We assume in this section that other firms do not respond to the actions of the entrant. The case of a leader-follower game will be analyzed in Section 4. The previous notation remains valid but subscripts 0 and 1 are used to denote the existing firms and the entrant, respectively.

The entrant is assumed to know the locations of the existing facilities. In effect, she can combine all of this information in a single step function as described in the previous section. However, we model the problem using the smooth approximation of the step function represented by $g_0(x)$. This way we maintain the continuity and avoid somewhat messy expressions. When fine-tuning an actual implementation, however, the step function should be used to achieve higher accuracy. As we have established in the previous section, given $g_0(x)$, the minimum distance and the maximum utility an arbitrary consumer x receives from the existing set of branches can be represented as random variables with the following distributions:

$$z_0(x) \sim \text{Uniform}[0, g(x)/2], \text{ and } u_0(x) \sim \text{Uniform } [a - bg_0(x)/2, a].$$

By the same token, given $g_1(x)$,

$$z_1(x) \sim \text{Uniform } [0, g_1(x)/2], \text{ and } u_1(x) \sim \text{Uniform } [a - bg_1(x)/2, a].$$

Here we make two assumptions: First, branches of both the entrant and the existing firms provide a very standardized product or service, i.e., a and b are the same for all firms (not only for branches that belong to a firm); second, the market is already fully served by the existing firms, i.e., $a - bg_0(x)/2 \geq 0$ for all $x \in [0, 1]$. Therefore, the entrant is fighting for a share in an already captured market. The first assumption can easily be relaxed albeit at the expense of analytical simplicity. The second assumption is well justified in mature markets. However, it could also be relaxed by dividing the market into two parts: Those that are captured and those that are not. Then we would employ the analysis presented in the previous section for the parts that are not captured.

The firm first decides whether to enter the market or not. She decides on $g_1(x)$ if the decision is affirmative. These decisions can easily be combined in the definition of the smooth function where $g_1(x) = \infty$ represents the no-entry decision in the part of the market that x belongs to. Let $P_1(x)$ denote the probability that the entrant captures consumer x. Then,

$$\begin{aligned} P_1(x) &= P\{u_1(x) \geq u_0(x)\} \\ &= P\{z_1(x) \leq z_0(x)\} \\ &= \int_0^{g_0(x)/2} P\{z_1(x) \leq z_0(x) | z_0(x) = z\} dF_{z_0(x)}(z) \\ &= \int_0^{g_0(x)/2} P\{z_1(x) \leq z\} \frac{2}{g_0(x)} dz \\ &= \begin{cases} \frac{g_0(x)}{2g_1(x)} & \text{if } g_1(x) \geq g_0(x) \\ 1 - \frac{g_1(x)}{2g_0(x)} & \text{if } g_1(x) \leq g_0(x). \end{cases} \end{aligned}$$

Hence, the entrant solves:

$$\max_{g_1(x) \geq 0} \Pi_1[g_1(x), g_0(x)] = \int_0^1 \left[P_1(x) f(x) - \frac{K_1(x)}{g_1(x)} \right] dx. \tag{7}$$

A function that maximizes the integral, maximizes the integrand at every point when the integrand does not contain any integral or derivative of the decision variables. This can easily be verified through Euler and Jacobi conditions in the calculus of variations (see for example, Connors and Teichroew 1967, pages 13 and 21). Therefore, we need to solve the following for each x:

$$\max_{g_1(x) \geq 0} \pi_1[g_1(x), g_0(x)] = P_1(x) f(x) - \frac{K_1(x)}{g_1(x)}.$$

There are two cases:

Case 1: $g_1(x) \geq g_0(x)$. Then the problem is:

$$\max_{g_1(x) \geq g_0(x)} \pi_1[g_1(x), g_0(x)] = \frac{g_0(x) f(x)}{2g_1(x)} - \frac{K_1(x)}{g_1(x)},$$

and the solution is:

$$g_1^*(x) = \begin{cases} g_0(x) & \text{if } g_0(x) f(x)/2 - K_1(x) \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Case 2: $g_1(x) \leq g_0(x)$. Then the problem is:

$$\max_{g_1(x) \leq g_0(x)} \pi_1[g_1(x), g_0(x)] = f(x) - \left(\frac{g_1(x) f(x)}{2g_0(x)} + \frac{K_1(x)}{g_1(x)} \right).$$

Let

$$\bar{g}(x) = \left(\frac{2g_0(x)K_1(x)}{f(x)}\right)^{1/2}.$$

If $\bar{g}(x) \leq g_0(x)$ then $g_0(x)f(x)/2 - K_1(x) \geq 0$, i.e., the same condition in as Case 1. Hence, we can summarize the entrant's best strategy as:

$$g_1^*(x) = \begin{cases} \min\{g_0(x), \bar{g}(x)\} & \text{if } g_0(x)f(x)/2 - K_1(x) \geq 0 \\ \infty & \text{otherwise.} \end{cases} \tag{8}$$

This is the main result of this section. It suggests that the firm will not enter parts of the market where it cannot recover its fixed costs. If the firm decides to enter a market (or a subset of the market) she will do so by opening at least as many stores as the existing competition has and hence, will capture at least half of the market. Parallel to a finding in the earlier section, we see that the firm enters the parts of the market where consumer density is above a certain threshold value, which now depends not only on the firm's fixed cost but also on the dispersion of the existing stores. However, consumer density has a slight but nonetheless direct effect on the firm's decision even beyond that threshold value (e.g., observe the impact of $f(x)$ on $\bar{g}(x)$). The same reasoning applies to the fixed cost. Hence, even a passive competition may force the firm to reconsider consumer density and her fixed cost. In general, however, once the entry conditions are satisfied, the firm's location strategy is somewhat robust to small changes in consumer density and fixed cost.

In the model, we have assumed that the firms offer a very standardized service. Therefore, we hold a and b constant for all branches regardless of their owners. It is straightforward to extend the analysis to different a and b's but the solution might not come out in closed form. In our analysis, we have not considered the possible economies of scale resulting from having a large presence in a region. Usually, firms benefit from economies of scale in the areas of regional management, distribution, advertisement, and work force training when they have a larger presence in a market. Our result suggests that even without those extensive economies of scale, there is an incentive for the firm to enter the market as strongly as the existing competition.

In the remainder of this section we will solve a simple illustrative example to demonstrate how our

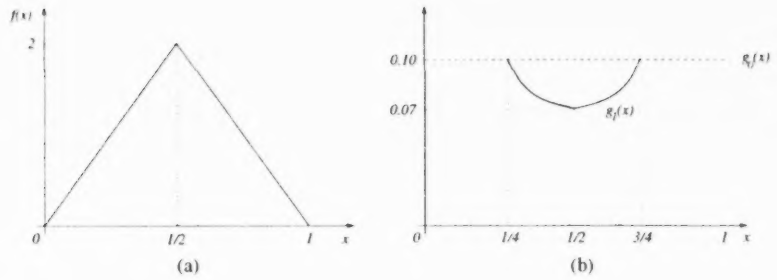

Figure 4: Density function (a) and the solution (b) for the example.

methodology works. To this end, consider a simple triangular consumer density given as:

$$f(x) = \begin{cases} 4x & \text{if } 0 \leq x \leq 1/2 \\ 4(1-x) & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

and depicted in Figure 4(a). Assume that there are ten existing stores equally spaced over the market, hence, $g_0(x) = 0.1$ for all $x \in [0,1]$. Also, let $K_1(x) = 0.05$ for all $x \in [0,1]$. Then the optimal location strategy of the entrant can be computed as:

$$g_1^*(x) = \begin{cases} \infty & \text{if } 0 \leq x < 1/4 \\ 0.05/\sqrt{x} & \text{if } 1/4 \leq x \leq 1/2 \\ 0.05/\sqrt{1-x} & \text{if } 1/2 \leq x \leq 3/4 \\ \infty & \text{if } 3/4 < x \leq 1. \end{cases}$$

Figure 4(b) depicts the solution of this example. A simple computation reveals that the average value of $g_1(x)$ and hence, the average distance between the neighbor branches is 0.083 in the interval [0.25, 0.75]. Therefore, the firm will open approximately six branches (i.e., 0.5/0.083=6.02) in the middle segment as compared to five existing branches. On the other hand, she will not open any stores in the first and fourth quartiles.

4 Duopoly: Stackelberg-Nash Equilibrium

We now extend the analysis to competition between two firms, where one of them has an advantage over the other. The advantage comes from being the first mover as well as having prior knowledge

of the subsequent arrival of a new competitor. For the sake of clarity, we assume that there are no existing stores that belong to other firms. Relaxing this assumption, however, is rather ad hoc. The game progresses as follows: First, the leader picks her locations. Then the follower observes the leader's choice and makes his decisions. He decides, first, whether to enter or not to enter, and then, picks where to locate the branches if the first decision is affirmative. We model the game assuming complete information. Hence, all of the cost and demand information is common knowledge to all parties. Denote the leader by subscript 0 and the follower by subscript 1. In this setting we assume that consumers patronize either one of the firms. The sequence of the game is as follows:

Stage 1: The leader chooses $g_0(x)$,

Stage 2: Having observed $g_0(x)$, the follower chooses $g_1(x)$.

In Stage 2, we do not explicitly model the entry decision because as described before $g_1(x) = \infty$ indicates the no-entry decision. The usual approach in this game is to start from Stage 2 and find the response function of the follower for any strategy the leader may choose. Hence, given $g_0(x)$ the optimal strategy of the follower is nothing but the solution of the entrant determined in the previous section, i.e., (8). Therefore, his best strategy is given by:

$$g_1^*(x) = \begin{cases} \min\left\{g_0(x), \left(\frac{2g_0(x)K_1(x)}{f(x)}\right)^{1/2}\right\} & \text{if } g_0(x)f(x)/2 - K_1(x) \geq 0 \\ \infty & \text{otherwise.} \end{cases}$$

Let us assume that the leader wants to deter entry. The most economical way for her to do so is to choose:

$$g_0^*(x) = \frac{2K_1(x)}{f(x)} - \epsilon, \tag{9}$$

where, ϵ is an arbitrarily small positive number. However, in order to simplify the notation, we avoid ϵ-strategies and assume that it is zero. Hence, the follower sets $g_1^*(x) = \infty$ for all $x \in [0,1]$ and the leader captures the entire market, i.e., $P_0(x) = 1$. Then, her total profit is given by:

$$\begin{aligned} \Pi_0[g_0^*(x), \infty] &= \int_0^1 \left[P_0(x)f(x) - \frac{K_0(x)}{g_0^*(x)}\right] dx \\ &= \int_0^1 \left[f(x) - f(x)\frac{K_0(x)}{2K_1(x)}\right] dx \\ &= \int_0^1 \left[1 - \frac{K_0(x)}{2K_1(x)}\right] f(x)dx. \end{aligned}$$

Hence, a sufficient condition for the leader to deter entry is to have

$$K_0(x) \leq 2K_1(x), \text{ for all } x \in [0,1], \tag{10}$$

so that her profit becomes positive. It should be noted that this condition is not necessary. There could be less restrictive cases where the leader can deter entry, depending on the consumer density and the values of fixed costs. Then, clearly, the leader incurs a loss in certain regions but covers those losses with the profits from other areas. This shows that a leader can retain her monopoly position even though she is considerably cost disadvantaged. As we have just shown, the leader's fixed costs could be more than twice the follower's, yet she could stay as a monopoly. Unlike the earlier findings, we observe that consumer density and fixed cost have a substantial impact on the strategy of a firm that wishes to stay as a monopoly against an entry threat. The impact of $f(x)$ on $g_0(x)$ is obvious from (9) but the impact of $K_0(x)$ should be interpreted in conjunction with condition (10).

A natural question is whether the leader should pursue such a strategy at all costs. Let us now consider the case where the leader allows entry, with the foreknowledge that she will lose at least half of the market to the follower. The leader sets $g_0(x)$ such that the follower enters the market (i.e., this is the case where $g_0(x)f(x)/2 - K_1(x) \geq 0$). Then,

$$\begin{aligned} g_1^*(x) &= \min\left\{g_0(x), \left(\frac{2g_0(x)K_1(x)}{f(x)}\right)^{1/2}\right\}, \\ &= \left(\frac{2g_0(x)K_1(x)}{f(x)}\right)^{1/2}, \end{aligned} \tag{11}$$

and

$$P_0(x) = \left(\frac{K_1(x)}{2f(x)g_0(x)}\right)^{1/2}.$$

Hence, the leader's problem is

$$\begin{aligned} \max_{g_0(x) \geq 2K_1(x)/f(x)} \Pi_0[g_0(x), g_1^*(x)] &= \int_0^1 \left[P_0(x)f(x) - \frac{K_0(x)}{g_0(x)}\right] dx, \\ &= \int_0^1 \left[f(x)\left(\frac{K_1(x)}{2f(x)g_0(x)}\right)^{1/2} - \frac{K_0(x)}{g_0(x)}\right] dx, \\ &= \int_0^1 \left[\left(\frac{K_1(x)f(x)}{2g_0(x)}\right)^{1/2} - \frac{K_0(x)}{g_0(x)}\right] dx. \end{aligned} \tag{12}$$

In the Appendix we show that the integrand is quasi-concave in $g_0(x)$ for all x. Therefore, the first order condition would provide the optimal solution given as:

$$g_0^*(x) = \left(\frac{8}{K_1(x)f(x)}\right)^{1/4} \sqrt{K_0(x)}, \tag{13}$$

provided $g_0^*(x) \geq 2K_1(x)/f(x)$. Hence, the follower may enter only if

$$\left(\frac{8}{K_1(x)f(x)}\right)^{1/4} \sqrt{K_0(x)} \geq 2K_1(x)/f(x),$$

i.e.,

$$K_0(x) \geq \sqrt{\frac{K_1(x)^5}{2f(x)^3}}. \tag{14}$$

The leader's profit is then:

$$\Pi_0[g_0^*(x), g_1^*(x)] = \int_0^1 \left(\frac{K_1(x)f(x)}{K_0(x)}\right)^{1/4} \left[\left(\frac{K_1(x)f(x)}{2}\right)^{3/8} \frac{1}{\sqrt{2}} - \left(\frac{K_0(x)}{2}\right)^{3/4}\right] dx$$

Then, a sufficient condition for the leader to have positive profit under entry is expressed as:

$$\sqrt[3]{2}K_0(x)^2 \leq K_1(x)f(x), \text{ for all } x \in [0,1], \tag{15}$$

i.e., the second term in the above integrand is positive. Now suppose $K_0(x) \leq 2K_1(x)$. Then the leader prefers to monopolize if

$$\left[1 - \frac{K_0(x)}{2K_1(x)}\right] f(x) \geq \left(\frac{K_1(x)f(x)}{K_0(x)}\right)^{1/4} \left[\left(\frac{K_1(x)f(x)}{2}\right)^{3/8} \frac{1}{\sqrt{2}} - \left(\frac{K_0(x)}{2}\right)^{3/4}\right]. \tag{16}$$

Therefore, depending on the values of $K_0(x), K_1(x)$, and $f(x)$, there could be regions where we see the leader, the follower, or the both in the market. Conditions (10), (14), (15), and (16) are depicted in Figure 5, where $\tilde{K}_0(x) = f(x)/2\sqrt[3]{2}$. Dotted lines denote the boundaries of the conditions while solid lines denote the active ones. Note that in some regions of parameters the leader prefers duopoly to monopoly.

It is clear that the leader has an undisputed first mover advantage. For example, she could monopolize the market even if she is considerably cost disadvantaged. Moreover, she could still retain some market and hence, make profit even when the advantage of the follower is severe. There is a certain region of fixed cost combinations where the leader does not enter the market. Hence, in order to have

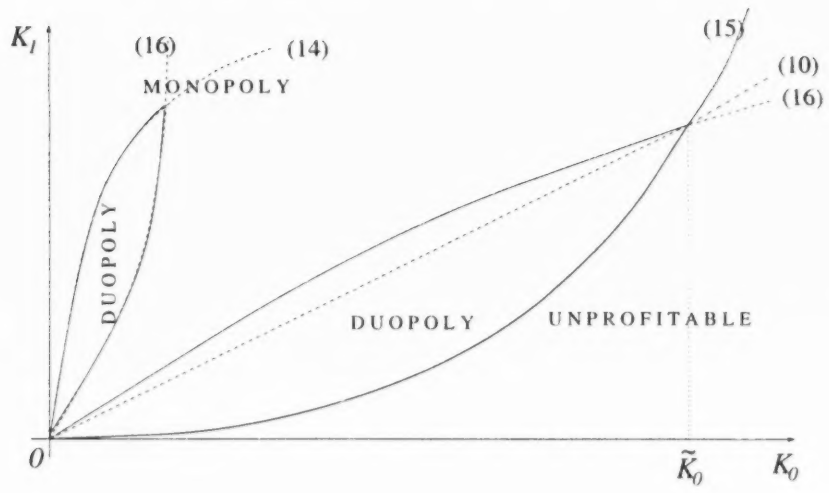

Figure 5: Range of fixed costs where the equilibrium output changes.

a game, condition (15) must be satisfied. Observe from the figure that in some cases, the leader has considerable advantage over the follower yet she prefers an entry by the follower rather than monopolization. However, we are unable to prove this for all values of $f(x)$ due to the intractability of condition (16).

Either by choice or out of necessity, once the leader lets the follower enter, her location strategy becomes quite insensitive to the changes in her fixed cost and more so to the changes in consumer density as seen in (13). The parameters, however, maintain the anticipated impact, i.e., the leader's stores become more dispersed in the areas where consumer density is low or her fixed cost is high. As a result of competition, the follower's fixed cost also has a slight impact on the leader's strategy. In the areas where the follower has high fixed cost, the leader's stores become less dispersed. When the follower's fixed cost reaches at a certain threshold, the leader may monopolize those areas by opening more stores.

If the follower enters the market, his location strategy can be found by substituting (13) into (11) as:

$$g_1^*(x) = \left(\frac{4K_1(x)}{f(x)}\right)^{3/8} \left(\sqrt{2}K_0(x)\right)^{1/4}.$$

Similar to that of the leader, the follower's location strategy is also insensitive to small changes in consumer density and fixed costs. As consumer density decreases or his fixed cost increases, the

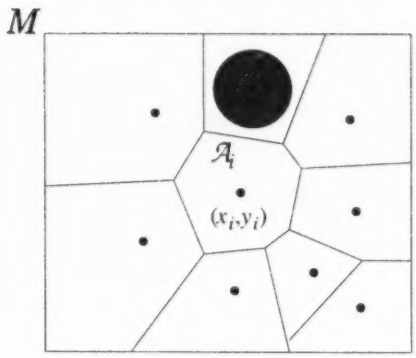

Figure 6: A market with branches ower a two-dimensional region.

follower's stores become less dispersed. However, as the leader's fixed cost increases, the follower's stores become *more* dispersed. This is in contrast to the leader's strategy and may seem intriguing at first. However, as the leader's fixed cost increases she will open fewer stores giving the follower a chance to have at least as much market share with fewer stores.

5 Competition in a Two-Dimensional Market

We now extend the analysis to a two-dimensional market. We show that our methodology works just as well and all the results derived in the earlier sections remain valid under a mild condition. We provide the extension only for the entrant's problem to demonstrate that the solutions in the one- and two-dimensional cases are identical. Although, we will carry out the analysis under the Euclidean distance assumption, the analysis can be adapted to other distance metrics.

Consider an arbitrary market $\mathcal{M}$ and a set of branches $(x_1, y_1), \ldots, (x_n, y_n)$ belonging to a firm. Assume that consumers are spread across the market according to a density $f(x, y)$. Given the branch locations and the distance metric, market areas $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of the branches can be characterized by an ordinary Voronoi diagram. An example configuration is depicted in Figure 6. However, whether a branch can capture all the consumers in its market areas depends on the size and the shape of the market area as well as the consumer utility. Similar to earlier definition, let

$$u_i(x, y) = a - b\sqrt{(x - x_i)^2 + (y - y_i)^2}$$

be the utility the consumer located at (x, y) derives from patronizing branch (x_i, y_i). It is clear that a consumer located at (x, y) achieves the maximum utility in the market area $\mathcal{A}_i$ that it belongs to because the branch of that market area is closest to that consumer. Let $z(x, y)$ denote the minimum distance from a consumer located at $(x, y) \in \mathcal{A}_i$ to a branch. The distribution of $z(x, y)$ can be approximated by:

$$P\{z(x, y) \leq z\} = \frac{\pi z^2}{A_i}, \text{ for } 0 \leq z \leq \sqrt{A_i/\pi},$$

where, A_i denote the area of $\mathcal{A}_i$. Note that the numerator is nothing but the area of a circle that has the branch location as the center and z as the radius. Hence, the probability is computed as the ratio of the area of the circle to the area of the market area (see Figure 6). As a word of caution, if another metric is used, then the area is no longer a circle. For example, under rectilinear metric the area will be a 45^o rotated square. We should note that this approximation is highly accurate if the market areas have closer to circular shapes (or rotated squares for rectilinear metric). We now follow the same steps as in the one-dimensional case. First, we define a step function, $A_s(x, y)$, over the market as:

$$A_s(x, y) = A_i, \text{ for all } (x, y) \in \mathcal{A}_i.$$

Next, we approximiate this step function with its continuous counterpart, $A(x, y)$. Figure 7 depicts the market areas and the corresponding step and smooth functions for an example configuration. Hence, the distribution of $z(x, y)$ can be approximated as:

$$P\{z(x, y) \leq z\} = \frac{\pi z^2}{A(x, y)}, \text{ for } 0 \leq z \leq \sqrt{A(x, y)/\pi}.$$

Note that, $A(x, y)$ serves the same purpose as $g(x)$ in the one-dimensional case. Their roles, however, are somewhat different. Here, $A(x, y)$ approximates the market area of each branch, whereas $g(x)$ approximates the distance between the branches. In each case, however, the firms express their location choices simply by these smooth functions. Note also that another difference between one- and two-dimensional cases is the distribution of the random variables that denote the minimum distance between a consumer and the stores (i.e., $z(x)$ and $z(x, y)$). In the one-dimensional case $z(x)$ has a uniform distribution whereas, in the two-dimensional case $z(x, y)$ has a triangular distribution.

Let $u_j(x, y)$ be the maximum utility consumer (x, y) derives by patronizing one of the branches belonging to Firm j and let $z_j(x, y)$ be the corresponding distance. Also let $A_j(x, y)$ denote the continuous

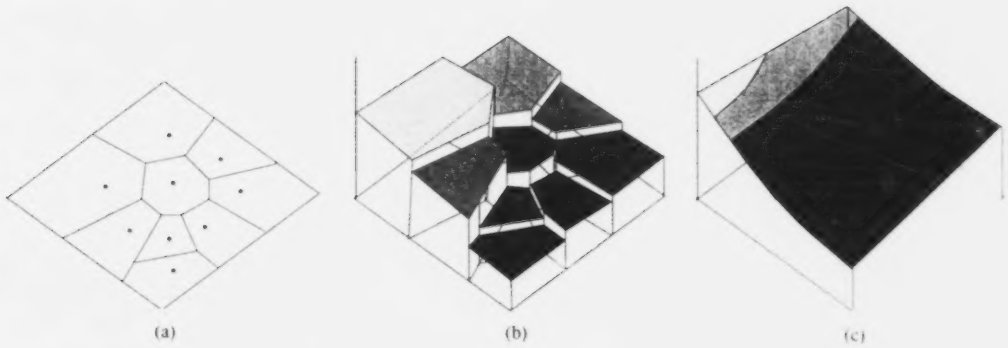

Figure 7: Market areas (a), the step function (b), and the smooth approximation (c) for an example.

function that approximates the market areas. We can compute the probability that the entrant captures the consumer at (x, y) as:

$$\begin{aligned}
P_1(x,y) &= P\{u_1(x,y) \geq u_0(x,y)\} \\
&= P\{z_1(x,y) \leq z_0(x,y)\} \\
&= \int_0^{\sqrt{A_0(x,y)/\pi}} P\{z_1(x,y) \leq z_0(x,y) | z_0(x,y) = z\} dF_{z_0(x,y)}(z) \\
&= \int_0^{\sqrt{A_0(x,y)/\pi}} P\{z_1(x,y) \leq z\} \frac{2\pi z}{A_0(x,y)} dz \\
&= \begin{cases} \frac{A_0(x,y)}{2A_1(x,y)} & \text{if } A_1(x,y) \geq A_0(x,y) \\ 1 - \frac{A_1(x,y)}{2A_0(x,y)} & \text{if } A_1(x,y) \leq A_0(x,y). \end{cases}
\end{aligned}$$

The treatment of fixed costs is essentially the same. Let $K_1(x, y)$ be the fixed cost of opening a branch at location (x, y), which is also assumed to be a slowly varying function of x and y. We repeat the earlier process as follows (we drop the index 1 only in this part for the sake of clarity):

$$\begin{aligned}
\sum_i K(x_i, y_i) &= \sum_i K(x_i, y_i) \int_{\mathcal{A}_i} \frac{1}{A_i} dxdy \\
&= \sum_j K(x_i, y_i) \int_{\mathcal{A}_i} \frac{1}{A_s(x,y)} dxdy \\
&\approx \sum_j \int_{\mathcal{A}_i} \frac{K(x,y)}{A_s(x,y)} dxdy \quad \text{(because } K(x,y) \text{ is slowly varying)} \\
&= \int_{\mathcal{M}} \frac{K(x,y)}{A_s(x,y)} dxdy \\
&\approx \int_{\mathcal{M}} \frac{K(x,y)}{A(x,y)} dxdy.
\end{aligned}$$

Hence, a profit-maximizing entrant solves the following problem:

$$\max_{A_1(x,y)\geq 0} \Pi_1[A_1(x,y), A_0(x,y)] = \int_{\mathcal{M}} \left[P_1(x,y) f(x,y) - \frac{K_1(x,y)}{A_1(x,y)} \right] dxdy. \tag{17}$$

Note that (17) is essentially the same as (7). Hence, we skip the rest of the analysis as it would be a replication of the earlier sections. Given our treatment it is fair to conclude that earlier results generalize to two-dimensional markets under a mild condition, i.e., market areas have shapes closer to their ideals under given distance metrics.

6 Concluding Remarks

We have discussed the problem of locating retail branches under competition. We have proposed a novel methodology that strikes a good balance between insightful economic models and detailed operations research models. Our models have produced several insightful and intuitive results without losing much of the information a decision maker might need. We have focused on the investigation of the impact of fixed costs on location patterns. We should note, however, that the models allow many other parametric analyses.

Our main results are general in one- and two-dimensional markets. We have observed that the leader has the first mover advantage, which she uses either to deter or to welcome entry. Furthermore, we have shown that the leader makes positive profits even if she is severely cost disadvantaged. On the other hand, when an entrant (or a follower) decides to enter the market, he does so by opening as many stores as the existing ones. We have established that the primary role of consumer density and fixed cost is on the market structure. It has been shown that the firms' entry and entry deterrence decisions are quite sensitive to changes in these parameters. The secondary role of consumer density and fixed costs is on the firms' location strategies once the market structure is defined. We have shown that these parameters have a small impact on the firms' strategies except for a firm that vigorously tries to maintain her monopoly position. However, we should note that under elastic demand conditions, these parameters would have a much higher impact.

Many of the assumptions we have made throughout the paper could be relaxed and handled by our

models, leading to future research avenues. First, elastic demand, more elaborate choice processes, random utilities, and consumer search patterns could be incorporated. Second, we have assumed common ownership of a firm as well as its branches. However, there are many cases where different ownerships and a variety of franchising contracts exist. Thus, including contracts such as those governing the exclusive territories would be an important contribution. Third, scale economies that are observed in the areas of regional management, distribution, advertisement, and work force training could also be incorporated. Finally, our models could be extended to include such store characteristics as service quality, merchandise assortment, and price.

Appendix

Here we prove that the integrand in the objective function (12) is quasi-concave. Let us write the integrand as

$$\pi(w) = \frac{A}{\sqrt{w}} - \frac{B}{w},$$

where, $w = g_0(x)$, $A = (K_1(x)/2f(x))^{1/2}$, and $B = K_0(x)$. Given w_1 and w_2, and $\lambda \in (0,1)$ we will prove that $\pi(w_0) \geq \min\{\pi(w_1), \pi(w_2)\}$, where, $w_0 = (1-\lambda)w_1 + \lambda w_2$. Rewrite the integrand as:

$$\pi(w) = \frac{A\sqrt{w} - B}{w}.$$

It is clear that the numerator is strictly concave. Thus

$$\begin{aligned}
\pi(w_0) &= \frac{A\sqrt{w_0} - B}{w_0} \\
&= \frac{A\sqrt{w_0} - B}{(1-\lambda)w_1 + \lambda w_2} \\
&\geq \frac{(1-\lambda)(A\sqrt{w_1} - B) + \lambda(A\sqrt{w_2} - B)}{(1-\lambda)w_1 + \lambda w_2} \\
&= \frac{(1-\lambda)w_1\pi(w_1) + \lambda w_2\pi(w_2)}{(1-\lambda)w_1 + \lambda w_2} \\
&= \frac{(1-\lambda)w_1}{(1-\lambda)w_1 + \lambda w_2}\pi(w_1) + \frac{\lambda w_2}{(1-\lambda)w_1 + \lambda w_2}\pi(w_2) \\
&\geq \min\{\pi(w_1), \pi(w_2)\}.\square
\end{aligned}$$

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References

[1] S.P. Anderson and A. de Palma. Multiproduct firms: A nested logit approach. *Journal of Industrial Economics*, 40:261–276, 1992.

[2] M. Aoyagi and A. Okabe. Spatial competition of firms in a two-dimensional bounded market. *Regional Science and Urban Economics*, 23:259–289, 1993.

[3] M. E. Ben-Akiva. *Structure of Passenger Travel Demand Models.* PhD thesis, Department of Civil Engineering, Massachusetts Institute of Technology, Cambridge, MA, 1973.

[4] A. Caplin and B. Nalebuff. Aggregation and imperfect competition: On the existence of equilibrium. *Econometrica*, 51:25–59, 1991.

[5] M.M. Connors and D. Teichroew. *Optimal Control of Dynamic Operations Research Models.* International Textbook Company, Scranton, PA, 1967.

[6] C.F. Daganzo. *Logistics Systems Analysis.* Springer-Verlag, Berlin, Heidelberg, 1991.

[7] C. d'Aspremont, J.J Gabszewicz, and J.-F. Thisse. On Hotelling's "Stability in competition". *Econometrica*, 47:1145–1150, 1979.

[8] P. Davis. Spatial competition in retail markets: Movie theatres. Technical report, MIT Sloan School of Management, 2000.

[9] A. de Palma, V. Ginsburgh, M. Labbé, and J.-F. Thisse. Competitive location with random utilities. *Transportation Science*, 23:244–252, 1989.

[10] G. Dobson and U.S. Karmarkar. Competitive location on a network. *Operations Research*, 35:565–574, 1987.

[11] B.C. Eaton. Spatial competition revisited. *Canadian Journal of Economics*, 5:268–277, 1972.

[12] H.A. Eiselt and G. Laporte. The $(1|x_p)$-medianoid problem on a network with edge demands. G-2000-61, Les Cahiers du GERAD, École des Hautes Études Commerciales, Montreal, 2001.

[13] H.A. Eiselt, G. Laporte, and J.-F. Thisse. Competitive location models: A framework and bibliography. *Transportation Science*, 27:44–54, 1993.

[14] P.B. Ellickson. Vertical product differentiation and concentration in the supermarket industry. Technical report, Simon School of Business Administration, University of Rochester, 2000.

[15] D. Erlenkotter. The general market area model. *Annals of Operations Research*, 18:45–70, 1989.

[16] J.J. Gabszewicz and J.-F. Thisse. Spatial competition and the location of firms. In R. Arnott, editor, *Location Theory*, pages 1–71. Harwood Academic Publishers, Chur, Switzerland, 1986.

[17] A.M. Geoffrion. The purpose of mathematical programming is insight, not numbers. *Interfaces*, 7(1):81–92, 1976.

[18] A.M. Geoffrion. Making better use of optimization capability in distribution system planning. *AIIE Transactions*, 11:96–108, 1979.

[19] A. Ghosh and S.L. McLafferty. *Location Strategies for Retail and Service Firms*. Lexington Books, Lexington, MA, 1987.

[20] S.L. Hakimi. On locating new facilities in a competitive environment. *European Journal of Operational Research*, 12:29–35, 1983.

[21] H. Hotelling. Stability in competition. *Economic Journal*, 39:41–57, 1929.

[22] D.L. Huff. Defining and estimating a trade area. *Journal of Marketing*, 28:34–38, 1964.

[23] D.L. Huff. A programmed solution for approximating an optimum retail location. *Land Economics*, 42:293–303, 1966.

[24] M. Kilkenny and J.-F. Thisse. Economics of location: A selective survey. *Computers & Operations Research*, 26:1369–1394, 1999.

[25] M. Labbé and S.L. Hakimi. Market and locational equilibrium for two competitors. *Operations Research*, 39:749–756, 1991.

[26] P.J. Lederer and A.P. Hurter. Competition of firms: Discriminatory pricing and location. *Econometrica*, 54:623–640, 1986.

[27] P.J. Lederer and J.-F. Thisse. Competitive location on networks under delivered pricing. *Operations Research Letters*, 9:147–153, 1990.

[28] D. McFadden. Conditional logit analysis of qualitative choice behavior. In P. Zarembka, editor, *Frontiers in Econometrics*. Academic Press, New York, 1973.

[29] T.C. Miller, T.L. Friesz, and R.L. Tobin. *Equilibrium Facility Location on Networks*. Springer-Verlag, Berlin, Heidelberg, New York, 1996.

[30] M. Nakanishi and L. Cooper. Parameter estimate for multiplicative interactive choice model: Least squares approach. *Journal of Marketing Research*, 11:303–311, 1974.

[31] G.F. Newell. Scheduling, location, transportation, and continuum mechanics: Some simple approximations to optimization problems. *SIAM Journal on Applied Mathematics*, 25:346–360, 1973.

[32] A. Okabe and M. Aoyagi. Existence of equilibrium configurations of competitive firms on an infinite two-dimensional space. *Journal of Urban Economics*, 29:349–370, 1991.

[33] A. Okabe and A. Suzuki. Stability of spatial competition of a large number of firms on an infinite two-dimensional space. *Environment and Planning A*, 19:1067–1082, 1987.

[34] F. Plastria. Static competitive facility location: An overview of optimisation approaches. *European Journal of Operational Research*, 129:461–470, 2001.

[35] C.S. ReVelle. The maximum capture or sphere of influence problem: Hotelling revisited on a network. *Journal of Regional Science*, 26:343–357, 1986.

[36] S. Salop. Monopolistic competition. *The Bell Journal of Economics*, 10:141–156, 1979.

[37] K. Seim. Spatial differentiation and firm entry: The video rental industry. Technical report, Department of Economics, Yale University, 2001.

[38] D. Serra, H.A. Eiselt, G. Laporte, and C.S. ReVelle. Market capture models under various customer-choice rules. *Environment and Planning B: Planning and Design*, 26:741–750, 1999.

[39] D. Serra and C.S. ReVelle. Market capture by two competitors: The preemptive capture problem. *Journal of Regional Science*, 34:549–561, 1994.

[40] A. Shaked and J. Sutton. Relaxing price competition through product differentiation. *Review of Economics Studies*, 49:3–13, 1982.

[41] L.A. Stole. Price discrimination in competitive markets. Graduate School of Business, University of Chicago, November 2001.